Sub - SPT   Set (A) CSE (IV sem)

Q-1 A and B take turn in throwing two die on the understanding that the first two throw 9 will be awarded prize. if A has first turn show that their chance of winning are in the ratio 9 : 8

Sol. Two die may have $6 \times 6 = 36$ ways

$p = 4/36 = 1/9 \quad q = 1 - p = 1 - 1/9 = 8/9$

Prob of 9 at first throw $= p$

" " " at Second $= qp$

" " " Third $= q^2 p$

Hence A's chance of winning = chance of 9 occurring all the first at same 1, 3, 5th. - -

$$P(A) = p + q^2 p + q^4 p + \cdots$$

$$= \frac{p}{1-q^2} \rightarrow (1)$$

Similarly

$$P(B) = qp + q^3 p + q^5 p + \cdots = \frac{qp}{1-q^2}$$

$$\frac{P(A)}{P(B)} = \frac{p}{qp} = \frac{1}{q} = 9/8$$

$\boxed{P(A) : P(B) = 9 : 8}$ Ans

or

Q-1 A Continuous random variable x that can be assume any value between $x = 1$ and $x = 4$ and is zero elsewhere has the density function given by $f(x) = \lambda(1+x)$. Find $P(X < 3) = ?$

Sol - 1

we know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$= \int_{-\infty}^{1} f(x)dx + \int_{1}^{4} f(x)\, dx + \int_{4}^{\infty} f(x)\, dx$$

$$= 0 + \int_{1}^{4} f(x)\, dx + 0 = \int_{1}^{4} \lambda(1+x)dx$$

$$= \left[\lambda x + \frac{\lambda x^2}{2}\right]_1^4 = 1$$

$$\lambda\left[3 + (8 - 1/2)\right] = 1 \Rightarrow (10 + 1/2)\lambda = 1$$

$$\lambda = \frac{2}{21}$$

$$F(3) = P(X \le 3) = \int_1^3 f(x)\, dx$$

$$= \frac{2}{21}\int_1^3 (1+x)\, dx = 2/21\left[x + \frac{x^2}{2}\right]_1^3$$

$$= \frac{2}{21}\left[2 + (9/2 - 1/2)\right] = 12/21 = 4/7 \text{ Ans}$$

Q.2 A random variable x has the prob distribution

X =	0	1	2	3	4	5	6	7
$P(X)$ =	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find K and $P\left(\frac{1.5 < X < 4.5}{X > 2}\right)$

Sol we have $\sum_{x=0}^{7} P(x) = 1$

$$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0 \qquad K = -1,\ K = 1/10$$

$K = -1$ (: neglect)

take $K = \frac{1}{10}$

$$P\left(\frac{1.5 < x < 4.5}{x > 2}\right) = \frac{P(2 < x < 4.5)}{P(x > 2)}$$

$$= \frac{P(X=3) + P(X=4)}{P(X>2)}$$

$$= \frac{2 \times \frac{1}{10} + \frac{3\times 1}{10}}{1 - [P(X=0) + P(X=1) + P(X=2)]}$$

$$= \frac{\frac{5}{10}}{1 - \left(\frac{3}{10}\right)} = 5/7 \text{ Ans}$$

Q2 A R.V x has the following density function

$x =$	-2	-1	0	1	2	3
$P(x) =$	0.1	K	0.2	$2K$	0.3	$3K$

Sol x is a random variate

$$\sum_{i=-2}^{3} P_i = 1$$

$$0.1 + 0.2 + 0.3 + 6K = 1$$

$$6K = 0.4$$

$$K = \frac{1}{15}$$

$$E(x) = \sum P_i x_i = (-2)(0.1) - K + 0 + 2K + 0.6 + 9K$$

$$= 10K + 0.4$$

$$\frac{10}{15} + \frac{4}{10}$$

$$= \frac{16}{15} \text{ Ans}$$

Q-3 Express the probability of poisson distribution its m.g.f moment about origin, central moment variance & S.D

Sol poisson distribution apply the following rules

(i) $p \to 0$ & $n \to$ very large.

prob of r successes $= \lim\limits_{\substack{p\to 0\, n\to\infty \\ np=m}} {}^{n}C_{r} p^{r} (1-p)^{n-r}$

$$P(x=r) = \lim_{\substack{n\to\infty \\ np=m}} \left\{ \frac{n(n-1)\cdots(n-\overline{r-1})}{\lfloor r} p^{r} \right\} \left(1-\frac{m}{n}\right)^{n} \left(1-\frac{m}{n}\right)^{-r}$$

Let $t = \left(1-\frac{m}{n}\right)^{n}$

$$\log(1-x) = -x - x^2/2 - x^3/3 - x^4/4 - \cdots$$

$$\log t = n\log\left(1-\frac{m}{n}\right) = n\left[-\frac{m}{n} - \frac{m^2}{2n^2} - \frac{m^3}{3n^3} - \cdots\right]$$

$$= -m - \frac{m^2}{2n} - \frac{m^3}{3n^2} - \cdots$$

$$t = e^{-m - m^2/2n - \frac{m^3}{3n^2} \cdots}$$

$$= e^{-m}$$

$$P(x=r) = \left[\lim_{n\to\infty} \frac{(np)^{r}}{\lfloor r}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)\right] e^{-m}$$

$$\boxed{P(x=r) = \frac{m^{r}}{\lfloor r} e^{-m}} \qquad r = 0, 1, 2, 3 \cdots$$

<u>m.g.f of poisson distribution:</u>

$$M_r(t) = E(e^{tr})$$

$$= \sum_{r=0}^{\infty} e^{-m} \frac{m^{r}}{\lfloor r} e^{tr} = \sum_{r=0}^{\infty} e^{-m} \frac{(me^{t})^{r}}{\lfloor r}$$

$$= e^{-m}\left(e^{me^{t}}\right)$$

we know $\quad \bar{r} = \mu_1' = \left[\frac{d}{dt} M_r(t)\right]_0 = \left[e^{-m}\left(e^{me^{t}}\right) me^{t}\right]_{t=0}$

$$= m$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_r(t)\right]_{t=0} = e^{-m}\left[\left(e^{me^t}(me^t)^2 + e^{me^t} me^t\right]_{t=0}\right.$$

$$= e^{-m} e^{m} [m^2+m] = m^2+m$$

$$\sigma^2 = \mu_2' - \mu_1'^2 = m^2 + m - m^2 = m$$

$$\boxed{S.D \text{ is } \sigma = \sqrt{m}}$$

Q-3 Define binomial distribution, m.g.f, mean, varia & S.D. ?

Sol Probability P_i of r success and $(n-r)$ failure in thes n trials is

$$P_i = P(X=r) = {}^nC_r\, q^{n-r} p^r$$

$$= \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r}$$

$$= {}^nC_0 q^n + {}^nC_1 q^{n-1} p + \cdots + {}^nC_n p^n$$

It is binomial expansion of $(q+p)^n$

$$\sum_{r=0}^{n} P(X=r) = \sum_{r=0}^{n} P_r = (q+p)^n = 1$$

m.g.f of binomial distribution -

M.g.f of binomial distribution about origin

= expected value of (e^{tr})

$$M_r(t) = E(e^{tr})$$

$$M_r(t) = \sum_{r=0}^{n} P_r e^{tr}$$

$$= \sum_{r=0}^{n} {}^nC_r q^{n-r} p^r e^{tr}$$

$$= \sum_{r=0}^{n} {}^nC_r q^{n-r} (pe^t)^r$$

$$= (q + pe^t)^n$$

mean $\mu_1' = \frac{d}{dt}[M_x(t)]$

$= [n(q+pe^t)^{n-1}pe^t]_{t=0}$

mean $= \mu_1' = np$

$\mu_2' = \left[\frac{d^2}{dt^2}M_x(t)\right] = [n(n-1)(q+pe^t)^{n-2}(pe^t)^2$
$+ n(q+pe^t)^{n-1}p \cdot e^t]_{t=0}$

$\mu_2' = n(n-1)p^2 + np$

$\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2$

$= n(n-1)p^2 + np - (np)^2$

$= -np^2 + np = np[1-p]$

$\boxed{\sigma^2 = npq}$ q

S.D $\quad \sigma = \sqrt{npq}$

Q-4 The sum of mean and variance of a binomial distribution is 15 and the sum of their square is 117 determine the distribution.

Sol we have mean $= np$
variance $= npq$

$np + npq = 15 \Rightarrow n^2p^2 + n^2p^2q^2 = 117$

$n^2p^2[1+q^2] = 117$

$\frac{(1+q)^2}{1+q^2} = \frac{225}{117} \Rightarrow 1 + \frac{2q}{1+q^2} = \frac{225}{117}$

$\frac{2q}{1+q^2} = \frac{12}{13}$

$\frac{q}{1+q^2} = \frac{6}{13}$

$6q^2-13q+6=0$

$$q=\frac{13\pm\sqrt{169-144}}{12}=\frac{13+5}{12} \text{ or } \frac{13-5}{12}$$

$q=2/3$ $np[1+q]=15$

$p=1-2/3=1/3$ $n=27$

Hence the required dis

$$P(X=r)={}^{27}C_r\left(2/3\right)^{27-r}\left(1/3\right)^r$$

$$r=0,1,2,\ldots 27$$

Q-4 Prob that a man aged 60 would be alive till the 70 years of age in 0.65. Find the probability that at least 7 out of 10 such men would be alive till 70 years of age

Sol $n=10$ $p=0.65$ $q=0.35$

$$P(X\geq 7)=P(7)+P(8)+P(9)+P(10)$$

$$={}^{10}C_7(0.35)^3(0.65)^7+{}^{10}C_8(0.35)^2\times(0.5)^8$$

$$+{}^{10}C_9(0.35)^1(0.65)^9+(0.65)^{10}$$

$$=120\times .00210+45\times .00390+10\times .00725$$

$$+0.01346$$

$$=.252+.1755+.0725+.01346$$

$$=0.513 \text{ Ans}$$

(B)

Q-1 Bay's theorem:

Sol Let $\{A_1, A_2, \dots A_n\}$ be an event space.
suppose an event A occurred but it is not
known which of the events $A_1, A_2 \dots A_n$ holds
true. then the conditional probability that event
A_K occurred give that A has been occurred
is given by

$$P\left(\frac{A_K}{A}\right) = \frac{P(A_K)\cdot P\left(\frac{A}{A_K}\right)}{\sum_{j=1}^{n} P(A_j)\cdot P\left(\frac{A}{A_j}\right)} \quad K=1,2,\dots n$$

Proof: Let $\{A_1, A_2 \dots A_n\}$ is be an event space
S so $A_1, A_2 \dots A_n$ are mutually exclusive and
exhaustive events

$$P(A_K \cdot A) = P(A)\cdot P\left(\frac{A_K}{A}\right)$$

$$= P(A_K)\cdot P\left(\frac{A}{A_K}\right)$$

$$\Rightarrow P\left(\frac{A_K}{A}\right) = \frac{P(A_K)\cdot P\left(\frac{A}{A_K}\right)}{P(A)}$$

But by theorem of total probability.

$$P(A) = \sum_{j=1}^{n} P(A_j)\cdot P\left(\frac{A}{A_j}\right)$$

$$P\left(\frac{A_K}{A}\right) = \frac{P(A_K)\cdot P\left(\frac{A}{A_K}\right)}{\sum_{j=1}^{n} P(A_j)\cdot P\left(\frac{A}{A_j}\right)}$$ Proved

$K=1,2,\dots n$

Q-1 The first four moment in a certain probability distribution about the point 4 are -1.5, 17, -30 and 108, calculate β_1 and β_2 and state whether the distribution is leptokurtic or platykurtic.

Sol Given

$$(\mu_1')_4 = \Sigma P_i(x_i - 4) = -1.5$$

$$\Sigma P_i x_i - 4 = -1.5$$

$$\bar{x} = 4 - 1.5 = 2.5$$

$$(\mu_2')_4 = \Sigma P_i (x_i - 4)^2 = 17$$

$$17 = \Sigma P_i [(x_i - \bar{x}) + (\bar{x} - 4)]^2$$

$$= \mu_2 + 0 + (2.5 - 4)^2$$

$$\mu_2 = 17 - 2.25 = 14.75$$

$$(\mu_3')_4 = \Sigma P_i (x_i - 4)^3 = -30$$

$$-30 = \Sigma P_i [(x_i - \bar{x}) + (\bar{x} - 4)]^3$$

$$= \mu_3 + 3(1.5)(14.75) + 0 - (1.5)^3$$

$$\mu_3 = -30 + 9/2 \times 59/4 + 3.375 = 39.75$$

$$(\mu_4)'_4 = \Sigma P_i (x_i - 4)^4 = 108$$

$$\mu_4 = 108 + 6(39.75) - 13.5(14.75) - (1.5)^4$$

$$= 142.3$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)} = 0.492$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.3}{(14.75)^2} = 0.65 < 3$$

so the curve is Platycurtic.

Q-2 If the life time of a component has pdf $\lambda e^{-\lambda t}$, $\lambda > 0$, $t > 0$ compute its time to failure, variance and failure rate function.

Sol Time to failure is its mean life time:

$$E(x) = \int_0^\infty t\, f(t)\,dt = \int_0^\infty \lambda t e^{-\lambda t}\,dt$$

$$= \lambda\left[\left(-\frac{t e^{-\lambda t}}{\lambda}\right)_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}\,dt\right]$$

$$= \lambda\left[0 - \frac{1}{\lambda^2}\left(e^{-\lambda t}\right)_0^\infty\right] = \frac{1}{\lambda}$$

$$\mu_2' = E(x^2) = \int_0^\infty t^2 f(t) = \lambda\int_0^\infty t^2 e^{-\lambda t}\,dt$$

$$= \lambda\left[\left(-\frac{t^2 e^{-\lambda t}}{\lambda}\right)_0^\infty + \frac{2}{\lambda}\int_0^\infty t e^{-\lambda t}\,dt\right]$$

$$\mu_2' = \frac{2}{\lambda^2}$$

Variance $\sigma^2 = \mu_2' - \mu_1'^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Failure rate function $\lambda(t) = \dfrac{f(t)}{R(t)}$

$f(t) = \lambda e^{-\lambda t}$ $\quad R(t) = 1 - F(t)$

$$= 1 - \int_0^t f(t)\,dt$$

$$= 1 - \lambda\int_0^t e^{-\lambda t}\,dt$$

$$= 1 - \left[\lambda\frac{e^{-\lambda t}}{\lambda}\right]_0^t = e^{-\lambda t}$$

$$\lambda(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \quad \underline{\text{Ans}}$$

Q-2 Given the joint probability density:

$$f(x,y) = \tfrac{2}{3}(x+2y) \qquad 0 < x < 1,\ 0 < y < 1$$
$$\qquad\quad 0 \qquad\qquad\quad \text{elsewhere.}$$

Find (i) Morginal density of X and Y

(2) Conditional density of X given $Y = y$

and use it evaluate $P\left[X \le 1/2 \mid Y = 1/2\right]$

Sol Morginal density of X is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_0^1 \tfrac{2}{3}(x+2y)\,dy =$$
$$= \begin{cases} \tfrac{2}{3}(x+1) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Similirly morginal density of y is

$$h(y) = \int_0^1 \tfrac{2}{3}(x+2y)\,dx$$
$$h(y) = \begin{cases} \tfrac{1}{3}(1+4y) & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(ii) By Conditional density:

$$\phi(x/y) = \frac{f(x,y)}{h(y)}$$

$$\phi(x/y) = \frac{\tfrac{2}{3}(x+2y)}{\tfrac{1}{3}(1+4y)} = \frac{2(x+2y)}{(1+4y)}$$

So conditional density of X given by $Y = y$ is

$$\phi(x/y) = \begin{cases} \dfrac{2(x+2y)}{1+4y} & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$$P\left[\frac{x \le 1/2}{y = 1/2}\right] = \int_0^{1/2} \frac{2(x+1)}{1+2}\,dx$$

$$= \tfrac{2}{3}\left[\frac{x^2}{2} + x\right] = \tfrac{1}{3}\left[\tfrac{1}{4} + 1\right] = \tfrac{5}{12} \text{ Ans}$$

Q-3 show that for the exponential distribution given by $dP = a e^{-x/c} dx$, $0 < x < \infty$, a being constant the mean and S.D are equal to c.

Sol: we have by p.d.f $\int_0^\infty dP = 1$

$$\int_0^\infty a e^{-x/c} dx = a(-c)\left[e^{-x/c}\right]_0^\infty$$

$$\Rightarrow ac = 1 \qquad \boxed{a = 1/c}$$

by def of Moment $\mu_r' = E(x^r) = \int_0^\infty x^r dP$

$$= \int_0^\infty x^r \, 1/c \, e^{-x/c} dx$$

$$= c^r \int_0^\infty v^r e^{-v} dx \qquad \text{if } x = cv, \; dx = c\,dv$$

$$= c^r \int_0^\infty v^{(r+1)-1} e^{-v} dv$$

$$= c^r \Gamma(r+1) = c^r \, L r \qquad \text{(by Gamma fun}$$

if $r = 1$ $\quad \mu_1' = c^1 \, L1 = c$

$$\mu_2' = c^2 \, L2 = 2c^2$$

$$\sigma^2 = 2c^2 - c^2 = c^2$$

$$S.D = \sqrt{c^2} = c \quad \text{proved}$$

Q-3 Find p.m.f of a random variate x whose prob generating function at $x = 3$

$$G_x(z) = \frac{2+z}{(2-z^2)(4-z)}$$

Sol. The P.G.F is given

$$G_x(z) = (2+z) \times \frac{1}{2}\left(1 - \frac{z^2}{2}\right)^{-1} \times \frac{1}{4}\left(1 - \frac{z}{4}\right)^{-1}$$

$$G_x(z) = \frac{1}{8}(2+z)\left[1 + \frac{z^2}{2} + \frac{z^4}{4} + \cdots\right]\left[1 + \frac{z}{4} + \frac{z^2}{16} + \frac{z^3}{64} + \cdots\right]$$

$$= \frac{1}{8}\left[(2+z)\left(1 + \frac{z^2}{2} + \frac{z^4}{4} + \cdots\right)\right]\left[1 + \frac{z}{4} + \frac{z^2}{16} + \frac{z^3}{64} + \cdots\right]$$

$$G_x(z) = 1/8\ [(2 + z + z^2 + z^3/3 + z^2/2 + z^5/4 + \cdots]$$
$$[1 + z/4 + z^2/16 + z^3/64 + \cdots]$$

prob at $x = 3$

$$P(x=3) = \frac{1}{4\times 64} + \frac{1}{8\times 16} + \frac{1}{8\times 4} + \frac{1}{8\times 2}$$
$$= \frac{1}{16}\left[\frac{1}{16} + 1/8 + 1/2 + 1\right] = 27/256 \text{ Ans}$$

Q-4 Find the m.g.f of the random variable whose p.m.f is given by $P(X=x) = 1/8\ {}^3C_x \quad x = 0, 1, 2, 3,$ and Find μ_1' μ_2' ?

Sol we have $M_x(t) = E(e^{tx})$

$$= \sum_{x=0}^{3} e^{tx} \cdot {}^3C_x\ 1/8$$
$$= 1/8\ [1 + 3e^t + 3e^{2t} + e^{3t}]$$
$$= 1/8\ [1 + e^t]^3$$

$$\mu_1' = \left[\frac{d}{dt}(M_x(t))\right] = 3/8\ [(1+e^t)^2 e^t]_{t=0}$$
$$= 3/2$$

$$\mu_2' = \frac{d^2}{dt^2}\left[M_x(t)\right] = 3/8\left[(1+e^t)^2 e^t + 3/4(1+e^t)\, e^{2t}\right]_{t=0}$$
$$= 3/2 + 3/2 = 3 \text{ Ans}$$

Q-4 Define exponential distribution, M.g.f, mean, variance & S.D

Sol A continuous variate X such that $0 < x < \infty$ is said to follow the exponential distribution when its prob density function is given by

$$f(x) = \begin{cases} \lambda e^{\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

we have $\int_0^\infty f(x)\,dx = \lambda \dfrac{[e^{-\lambda x}]_0^\infty}{-\lambda} = 1$

M.g.f : $M_X(t) = E[e^{tX}] = \int_0^\infty f(x)\, e^{tx} dx$

$$= \lambda \int_0^\infty e^{-(\lambda - t)} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda - t)} \right]_0^\infty = \frac{\lambda(0-1)}{-(\lambda - t)} = \frac{\lambda}{\lambda - t}$$

$$= [1 - t/\lambda]^{-1} = 1 + 1/\lambda\, t + 1/\lambda^2\, t^2 + \cdots$$

Mean $\bar{x} = \mu_1' = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0} = \left[\frac{\lambda}{(\lambda - t)^2}\right]_{t=0}$

$$= \lambda/\lambda^2 = 1/\lambda$$

Variance - $\sigma^2 = \mu_2' - (\mu_1')^2$

$$= \left[\frac{2\lambda}{(\lambda - t)^3}\right]_{t=0} - 1/\lambda^2$$

$$\sigma^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$$

$$S.D = \sigma = \sqrt{1/\lambda^2} = 1/\lambda \quad \text{Ans}$$

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I Mid Term examination
Session: 2017-18
B.Tech II Year CSE (IV Semester)
Subject with code: SPT
SET-B

Time: 2hrs. M.M :20

Attempt all questions

Q-1 Prove that Baye"s theorem !

OR

The first four moment in a certain probability distribution about the point 4 are -1.5, 17, -30 and 108. Calculate β_1, β_2, and state whether the distribution is lepto kurtic or platykurtic.

Q-2 If the life time of a component has probability density function $\lambda e^{-\lambda t}$. $\lambda > 0$, t>o. compute its time to failure, variance and failure rate function.

OR

Given the joint probability density .

f(x,y)= $\frac{2}{3}(x+2y)$ 0<x<1, 0<y<1

0 Elsewhere

Find (1) marginal density of X and Y.

(2) Conditional density of X given Y=y. and use it to evaluate $P(X \leq 1/2|Y = 1/2)$.

Q-3 Show that for the exponential distribution given by dp=ae$^{-x/c}$ dx, and x lies between 0 to infinite , a being a constant, the mean standard deviation are equal to c ..

OR

Find the probability mass function of a random variate x whose probability generating function is

$G_X(Z) = \frac{2+Z}{(2-Z^2)(4-Z)}$ at X=3 .

Q-4 Find the m.g.f of the random variable X whose p.m.f is given by

P(X=x) =1/8 C^3_x ; x=0 , 1, 2, 3 and then find μ_1' μ_2' !

OR

Define exponential distribution and its MGF, mean, variance, SD