Solution of DMS

Set-A

Rajasthan Institute of Engineering & Technology, Jaipur

Branch: - **Computer Science Engineering (CSE)**

Subject: - **Discrete Mathematical Structures (DMS)**

Q.1 (a) Explain the Addition Principle theorem with suitable example.

For two finite sets A and B which are disjoint, prove that

n(A U B) = n(A) + n(B)

Sol. (1) (a). Theorem – For two finite sets A and B which are disjoint , prove that

n(AUB) = n(A) + n(B)

Proof: - Let A have m1 elements than n(A)=m1 and Let B have m2 elements than n(A)=m2.

Since A and B are disjoint (having no elements in common) therefore AUB will have all the elements of A and all the elements of B.

So, numbers of elements in AUB is m1+ m2.

n(AUB) = m1+ m2

n(AUB) = n(A) + n(B)

Q.1 (b) Define the Function and Explain the Domain and Co-Domain and Range with suitable example.

Sol. (1) (b).

Function - A function is a relationship between two sets of numbers. We may think of this as a *mapping*; a function *maps* a number in one set to a number in another set. Notice that a function maps values to **one and only one** value. Two values in one set could map to one value, but one value **must never** map to two values: that would be a relation, *not* a function.

A function f is commonly declared by stating its domain X and codomain Y using the expression

$$\mathbf{f : X \rightarrow Y}$$

Example:

f(x) = x/2 ("f of x is x divided by 2") is a function, because each input "x" has a single output "x/2":

- f(2) = 1
- f(16) = 8
- f(−10) = −5

Definition of the Domain of a Function

For a function f defined by an expression with variable x, the implied domain of f is the set of all real numbers variable x can take such that the expression defining the function is real. The domain can also be given explicitly.

Definition of the Range of a Function

The range of f is the set of all values that the function takes when x takes values in the domain.

Definition of the Co-domain of a Function

The Codomain and Range are both on the output side, but are subtly different.

The Codomain is the set of values that could **possibly** come out. The Codomain is actually **part of the definition** of the function.

Example

A = {1, 2, 3, 4}

B= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
f(x) = 2x+1

• The set "A" is the **Domain**,
• The set "B" is the **Codomain**,
• And the set of elements that get pointed to in B (the actual values produced by the function) are the **Range**, also called the Image.
And we have:

- Domain: {1, 2, 3, 4}
- Codomain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
- Range: {3, 5, 7, 9}

OR

Q.1 (a) Explain the Pigeonhole Principle Theorem and also Generalized Pigeonhole Principle Theorem

Sol. (1) (a).

Pigeonhole Principle - Suppose you have k-pigeonholes and n-pigeons to be placed in them. If n > k(# pigeons># pigeonholes) then at least one pigeonhole contains at least two pigeons. In problem solving, the "pigeons" are often numbers or objects, and the "pigeonholes" are properties that the numbers/objects might possess.

Pigeonhole Principle Theorem - If "*n*" number of pigeons or objects are to placed in "*k*" number of pigeonholes or boxes; where $k<n$, then there must be at least one pigeonhole or box which has more than one object.

If k is a positive integer and k + 1 objects are placed into k boxes, then at least one of the boxes will contain two ore more objects

Proof : Suppose on the contrary that the proposition is false. Then, we have the case that
(i) k + 1 objects are placed into k boxes, and
(ii) no boxes contain two or more objects. From (ii), it follows that the total number of objects is at most k(since each box has 0 or 1 objects). Thus, a contradiction occurs

Generalized Pigeonhole Principle Theorem – If n-pigeons are sitting in k pigeonholes, where n > k, then there is at least one pigeonhole with at least n/k pigeons.

If k is a positive integer and N objects are placed into k boxes, then at least one of the boxes will contain N/k or more objects.
Here, $\lceil x \rceil$ is called the ceiling function, which represents the round-up value of x

If N objects are placed into k boxes, then there is at least one box containing at least ⌈N/k⌉ objects.

Proof by contradiction: Suppose that none of the boxes contains more than ⌈N/k⌉ objects. Then, the total number of objects is at most ⌈N/k⌉-1 objects.

$$\lceil N/k \rceil < (N/k)+1$$
$$k\,(\lceil N/k \rceil - 1) < k[(\lceil N/k \rceil + 1) - 1] = N$$

This is a contradiction because there are a total of N objects.

I don't understand how that inequality shows it's a contradiction, how did they get that the inequality shows less than N objects?

Q.1 (b) Define the Constant Function and Proof the following Theorem

The logarithmic function f(x)= log $_a$(x) is the inverse of the exponential function g(x) = a^x

Sol. (1) (b).

⇒ Constant function :- The function $f: A \to B$ such that $f(a) = c$; $\forall a \in A$ and $c \in B$ is called a constant function

In other words, $f: A \to B$ is a constant function if the range of f contains exactly one element.

e.g. if $A = \{\pi, 2\pi, 3\pi\}$ and $f(x) = \sin x$, $x \in A$ then f is constant. [Since $\sin x = 0$, $\forall x \in A$].

Theorem – The logarithmic function $f(x) = \log_a x$ is the inverse of the exponential function $g(x) = a^x$

Proof – Since $a^{\log_a x} = x$, $\forall x > 0$

and $\log_a(a^x) = x$, $\forall x \in R$

Now $(fog)(x) = f[g(x)] = f[a^x] = \log_a(a^x) = x$

$(gof)(x) = g[f(x)] = g[\log_a x] = a^{\log_a x} = x$

So, $fog(x) = x = (gof)x$.

Thus fog and gof are identity functions.

then f and g are the inverse of each other.

Q.2 (a) Define the Intersection operation of the Set and define also its properties with suitable example.

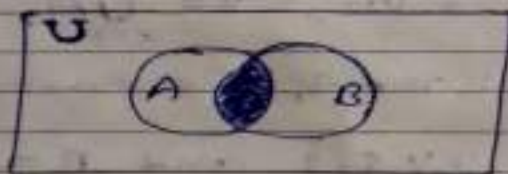
Sol. (**2**) **(a).**

2> Intersection Operation :- The intersection of two sets A and B, denoted by $A \cap B$, is defined as the set containing those elements which belong to A and B both.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \cap B$ read as "A intersection B"

Venn diagram of $A \cap B$.



The intersection of a finite number of the set $A_1, A_2, \ldots A_n$ is denoted by

$\bigcap_{r=1}^{n} A_r = A_1 \cap A_2 \ldots \cap A_n$

$\bigcap_{r=1}^{n} A_r = \{x : x \in A_r, 1 \le r \le n\}$

eg. if $A = \{x : x$ is multiple of $3\} = \{3, 6, 9, 12, 15, 18, \ldots\}$

$B = \{x : x$ is multiple of $4\} = \{4, 8, 12, 16, 20, 24, \ldots\}$.

$A \cap B = \{x : x$ is a common multiple of 3 and 4$\}$

$A \cap B = \{12, 24, 36, 48, 60, \ldots\}$

→ Properties of Intersection Operation -

1> $A \cap A = A$
2> $A \cap \phi = \phi$
3> $A \cap U = A$
4> $A \cap B = B \cap A$
5> $(A \cap B) \cap C = A \cap (B \cap C)$.

Q.2 (b) What is the Set and define the Complement of a set and also define the properties with suitable example.

Sol. (2) (b). The present definition of a set may sound very vague. A **set** can be defined as a unordered collection of **entities** that are related because they obey a certain **rule**.
'Entities' may be anything, *literally*: numbers, people, shapes, cities, bits of text, ... etc
The key fact about the 'rule' they all obey is that it must be *well-defined*. In other words, it must describe ***clearly*** what the entities obey. If the entities we're talking about are words, for example, a well-defined rule is: X is English
The set of all elements of U (universal set) that are not elements of $A \subseteq U$ is called the complement of A. The complement of A is
denoted by A'

$A' = \{x : x \in U \text{ and } x \text{ not } \in A\}$

For example,
Let U = $\{a,b,c,d,e,f,g,h\}$ and $A = \{b,d,g,h\}$.
Then $A' = \{a,c,e,f\}$
In Venn diagram A', the complement of set A

OR

Q.2 (a) What is Distributive Properties and explain and proof the following rules with suitable example.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Sol. (2) (a).

1> Distributive property of union over intersection

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof :-

$x \in A \cup (B \cap C) \Rightarrow [x \in A]$ or $[x \in (B \cap C)]$

$\Rightarrow$

Let $x \in A$. Then we

$x \in A \Rightarrow x \in A \cup B$

and $x \in A \Rightarrow x \in A \cup C$

therefore $x \in A \Rightarrow (x \in A \cup B)$ and $x \in (A \cup C)$.

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$.

Q.2 (b) What is Set and define the Subset and its properties and Set S – T (Difference of Two Sets) with suitable example.

Sol. (2) (b).

Set - A set is a collection of well-defined objects. For a collection to be a set it is necessary that it Should be well defined.
If a collection is a set then each object of this collection is said to be an element of this set. A set is usually denoted by capital letters of English alphabet and its elements are denoted by small Letters.

For example, A = Toy elephant, packet of sweets, magazines.
A set is represented by listing all its elements, separating these by commas and Enclosing these in curly bracket.
If V be the set of vowels of English alphabet, it can be written in Roster form as :
V = { a, e, i, o, u}

In this form elements of the set are not listed but these are represented by some common Property.

Let V be the set of vowels of English alphabet then V can be written in the set builder form as:

V = {x : x is a vowel of English alphabet}

→ Set S – T (Difference of two sets)

for two sets S and T, set S – T is defined as

$S - T = \{x : x \in S \text{ but } x \notin T\}$

$T - S = \{x : x \in T \text{ but } x \notin S\}$

eg. If $S = \{1, 2, 3, 4, 5, 6\}$ and $T = \{5, 6, 7, 8, 9, 10\}$

then $S - T = \{1, 2, 3, 4\}$ and $T - S = \{7, 8, 9, 10\}$

Q.3 (a) Explain the Matrix Representation of graph.

Find the Adjacency Matrix of the following relation.

X={ 1 , 2 , 3 } , Y={ 1 , 2 } and R={ (2,1) , (3,1) , (3,2) }.

Sol. (3) (a).

⇒ Adjacency Matrix of a Relation :–

Let elements of sets A and B be as

$A = \{a_1, a_2, \ldots\ldots a_m\}$

$B = \{b_1, b_2, \ldots\ldots b_n\}$

$R : A \to B$ then matrix of Relation is

$$M_R = [m_{ij}]_{m \times n} \qquad m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

M_R = matrix of Relation R or adjacency matrix of Relation

eg. Let $X = \{1, 2, 3\}$, $Y = \{1, 2\}$ and $R = \{(2,1), (3,1), (3,2)\}$ then M_R is a 3×2 matrix, whose elements m_{21}, m_{31}, m_{32} are each equal to unity and other elements are all zero.

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

eg. Find the relation R, when

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$R : X \to Y$ when $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3, y_4, y_5\}$

Sol^n then the relation is –

$R = \{(x_1, y_2), (x_1, y_3), (x_1, y_5), (x_2, y_1), (x_2, y_3)\}$.

Q.3 (b) Define the Join and Meet of Boolean Matrices. Also Perform the Join and Meet operation of following Two Boolean Matrices.

Sol. (3) (b).

→ Join and Meet of Boolean Matrices –

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two Boolean matrices of same size.

The join of A and B, denoted by $A \vee B$ is a matrix $C = [c_{ij}]_{m\times n}$

$$c_{ij} = \begin{cases} 1, & \text{if } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0, & \text{if } a_{ij} = 0 = b_{ij} \end{cases}$$

$[A \vee B = C]$

The meet of A and B, denoted by $A \wedge B$, is a matrix $D = [d_{ij}]_{m\times n}$ where

$$d_{ij} = \begin{cases} 1, & \text{if } a_{ij} = 1 = b_{ij} \\ 0, & \text{if } a_{ij} = 0 \text{ or } b_{ij} = 0 \end{cases}$$

$[A \wedge B = D]$

* The join and meet of two Boolean matrix are also Boolean matrices.

eg. Compute the join and meet of the matrices.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$C = A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = A \wedge B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

OR

Q.3 Explain the Recursive definition of a Set with suitable example. Also define Power Set with suitable example.

Sol. (3)

→ Recursive Definition of A set :-

An objects can be defined in terms of itself. This process is called Recursive.

A recursive definition of a set S consists of three clauses:

1.> Basis clause – explicitly lists atleast one element in S.

2.> Recursive clause – provides rules used to generate new elements of the set S from the known elements.

3.> Terminal clause – ensures that the first two clause are the only means to determine the elements of S.

→ Power Set :- The set of all subsets of a given set S is called the power set of S and is denoted by P(S). i.e.

$P(S) = \{T : T \subseteq S\}$

ϕ and S are both the member of P(S).

e.g. Let $S = \{a, b\}$ then $P(S) = \{\{a\}, \{b\}, \{a,b\}, \phi\}$.

Q.4 Define the following functions with examples:

Q.4 (i) Floor and Ceiling functions

Sol. (4) (i).

→ 5.) Floor Function or Greatest Integer Function

Let R be the set of real number, then the function $f : R \to Z$ is called the floor function denoted by $\lfloor x \rfloor$, if $f(x) = \lfloor x \rfloor$ = the largest integer that is less than or equal to x.

e.g. $\lfloor 7.2 \rfloor = 7$; $\lfloor -9.2 \rfloor = -10$; $\lfloor 5 \rfloor = 5$.

→ 6.) Ceiling Function :-

A function $f : R \to Z$ is said to be the ceiling function, denoted by $\lceil x \rceil$, if $g(x) = \lceil x \rceil$ = the smallest integer that is greater than or equal to x.

e.g. $\lceil 8.5 \rceil = 9$, $\lceil -5.3 \rceil = -5$; $\lceil 17 \rceil = 17$.

Theorem - Let x be any real number and n any integer. Then

i) $\lfloor n \rfloor = n = \lceil n \rceil$ ii) $\lceil x \rceil = \lfloor x \rfloor + 1$ $(x \notin Z)$

iii) $\lceil x+n \rceil = \lceil x \rceil + n$ iv) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

v) $\lfloor n/2 \rfloor = \frac{n-1}{2}$, if n is odd.

vi) $\lceil n/2 \rceil = \frac{n+1}{2}$, if n is odd

Q.4 (ii) Div and Mod functions.

Sol. (4) (ii).

→8> Div Function :-

The div function x div y. denoted by div(x,y), is the quotient when x is divided by y the remainder should be non-negative.

eg. div(22,5) = 22 div 5 = 4 ;

div(5,6) = 0.

we can define the other operators in terms of the div operator as follow.

i> Floor function $\lfloor \frac{a}{b} \rfloor$ = div(a,b).

eg. div(23,5) = 4 and $\lfloor \frac{23}{5} \rfloor = 4$.

ii> Ceiling function $\lceil \frac{a}{b} \rceil$ = − div(−a, b)

eg. $\lceil \frac{23}{5} \rceil$ = 5 and

− div(−23, 5) = −[quotient when (−25 + 2) is divided by 5 (to make remainder non-negative)]

= −(−5) = 5.

→7> Mod Function :-

The mod function mod(x,y) = x mod y denotes the remainder when an integer x is divided by a positive integer y.

eg. 24 mod 5 = 4.

eg. mod(15,3) = 15 mod 3 = 0

★ The mod function can be used to determine the day of the week in n days from a given particular day.

eg. Today is Tuesday, what day of the week will it be in 100 days from today ?

Sol^n :- n = 100, so

x = 100 mod 7 = 2

2nd day from today (i.e. Tuesday) is Thursday. Thus it will be Thursday in 100 days from Tuesday.

OR

Q.4 Define the following terms with examples:

(i) Equal set and Universal set.

Sol. (4)(i)

Equal sets :- When two sets S and T contain consists of the same element, they are called equal and we shall write

S = T. if S and T are not equal, we shall write S ≠ T.

eg. when S = {Ram, Shyam, Mohan}

T = {Mohan, Shyam, Ram}

then S = T.

eg. when S = {1, 2, 3} and T = {2, 1, 1, 2, 2, 3, 1, 3}

then S = T.

eg. when S = {1, 2, 3}, and T = {0, 1, 2, 3}

S ≠ T. since 0 is an element of T but not of S.

→ **Universal Set** :- If all the sets under consideration are subsets of a fixed set, then this fixed set is called universal set and denoted by **U**.

eg. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$, $S = \{1, 3, 5, 7\}$, $T = \{2, 4, 6, 8, \ldots\}$.
then U is the Universal set for S and T.

Q.4 (ii) Finite set and Infinite set.

Sol. (4)(ii)

→ **Finite Set** :- A set is said to be a finite set if the number of its distinct element is finite.

eg. Set $S = \{x : x \in N \text{ and } x \leq 10\}$ is a finite set

→ **Infinite Set** :- A set having infinite number of distinct elements is called the infinite set.

eg. Set $S = \{x \mid x \in N \text{ and } x \geq 10\}$ is an infinite set.

Solution of DMS

Set-B

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Q.1 (a) Explain the Addition Principle theorem with suitable example.

For two finite sets A and B, prove that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Sol. (1)(a)

⇒ Addition Principle :-

Theorem - for two finite sets A and B which are disjoint, prove that
$n(A \cup B) = n(A) + n(B)$.

Proof :-
Let A have m_1 elements than $n(A) = m_1$ and B have m_2 elements than $n(B) = m_2$

Since A and B are disjoint (having no elements in common) therefore $A \cup B$ will have all the elements of A and all the elements of B. So number of elements in $A \cup B$ is $m_1 + m_2$.
So,

$n(A \cup B) = m_1 + m_2$
$n(A \cup B) = n(A) + n(B)$. Proved.

Theorem - Prove, for finite sets A and B.
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
proof - we know that
$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B$ —— ①
and $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint, therefore.
$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$ — ②
further $A = (A - B) \cup (A \cap B)$.
and $(A - B) \cap (A \cap B) = \phi$

So, $n(A) = n(A - B) + n(A \cap B)$ —— ③
like $n(B) = n(A \cap B) + n(B - A)$ —— ④

adding eq^n (3) and (4).

$n(A) + n(B) = \{n(A-B) + n(A \cap B) + n(B-A)\} + \{n(A \cap B)\}$

$\Rightarrow n(A \cup B) + n(A \cap B)$ [using eq^n (2)]

thus.

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

$|A \cup B| = |A| + |B| - |A \cap B|$

that is the cardinality of $A \cup B$ is derived by including the cardinality of A and B both and excluding the cardinality of $A \cap B$. then the addition principle is also called as Principle of Inclusion and Exclusion.

Q.1 (b) Explain the Generalized Pigeonhole Principle Theorem With suitable example.

Sol. (1)(b)

Generalized Pigeonhole Principle Theorem – If n-pigeons are sitting in k pigeonholes, where n > k, then there is at least one pigeonhole with at least n/k pigeons. If k is a positive integer and N objects are placed into k boxes, then at least one of the boxes will contain N/k or more objects. Here, $\lceil x \rceil$ is called the ceiling function, which represents the round-up value of x**.** If N objects are placed into k boxes, then there is at least one box containing at least ⌈N/k⌉ objects.

Proof by contradiction: Suppose that none of the boxes contains more than ⌈N/k⌉ objects. Then, the total number of objects is at most ⌈N/k⌉-1 objects.

$$\lceil N/k \rceil < (N/k)+1$$
$$k\,(\lceil N/k \rceil - 1) < k[(\lceil N/k \rceil + 1) - 1] = N$$

This is a contradiction because there are a total of N objects.

I don't understand how that inequality shows it's a contradiction, how did they get that the inequality shows less than N objects?

eg. Let 5 separate departments in a department store and total no. of employee are 36. Show that one of the departments must have at least 8 employee.

Soln :-

36 employee are pigeons and 5 department are pigeonholes, then principle of pigeonhole is —

$\left\lfloor \frac{36-1}{5} \right\rfloor + 1$ employees

$\Rightarrow \left\lfloor \frac{35}{5} \right\rfloor + 1 = 7 + 1 = 8$.

OR

Q.1 (a) Define Constant Function, Equal Function and Identity Function with suitable example.

Sol. (1)(a)

Constant Function -

⇒ Constant function :- The function $f: A \to B$ such that $f(a) = c$; $\forall a \in A$ and $c \in B$ is called a constant function

In other words, $f: A \to B$ is a constant function if the range of f contains exactly one element.

eg: if $A = \{\pi, 2\pi, 3\pi\}$ and $f(x) = \sin x$, $x \in A$ then f is constant. [Since $\sin x = 0$, $\forall x \in A$].

Equal Function –

⇒ Equal function :-
Two functions f and g are said to be equal iff –

1.) the domain of f = the domain of g.

2.) the co-domain of f = the co-domain of g.

3.) $f(x) = g(x)$ for every x belonging to their common domain.

eg. Let $A = \{1, 2\}$, $B = \{3, 6\}$

$f: A \to B$; $f(x) = x^2 + 2$

$g: A \to B$; $g(x) = 3x$

$$\left.\begin{aligned} f(1) &= 3 = g(1) \\ f(2) &= 6 = g(2) \end{aligned}\right\} \Rightarrow f = g$$

Identity Function -

⇒ Identity Function :-

A function $f : A \to A$ is said to be an identity function of A iff f associates every element of A to the element itself.

$f : A \to A$ is an identity iff $f(x) = x$, $\forall x \in A$.

Q.1 (b) Define the following function with suitable example.

(i) Polynomial Function

Sol. (1)(b)(i)

Polynomial Function-

A function $f : R \to R$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where $a_i \in R$, $i = 1$ to n and $a_n \neq 0$ is called a polynomial function of degree n.

for $n = 1$, $f(x) = a_1 x + a_0$ (linear function)

for $n = 2$, $f(x) = a_2 x^2 + a_1 x + a_0$ (quadratic function).

Q.1 (b) (ii) Exponential Function

Sol. (1)(b)(ii)

Exponential Function –

⇒ Equal Function :-

Two functions f and g are said to be equal iff –

1.> the domain of f = the domain of g.

2.> the co-domain of f = the co-domain of g.

3.> $f(x) = g(x)$ for every x belonging to their common domain.

eg. Let $A = \{1, 2\}$, $B = \{3, 6\}$

$f : A \to B$; $f(x) = x^2 + 2$

$g : A \to B$; $g(x) = 3x$

$$\left.\begin{array}{l} f(1) = 3 = g(1) \\ f(2) = 6 = g(2) \end{array}\right\} \Rightarrow f = g$$

→ Properties of Exponential function :-

The exponential function $f(x) = a^x$

1.> dom $(f) = (-\infty, \infty)$; range $(f) = (0, \infty)$.

2.> The y-intercept of the graph is 1.

3.> if $a > 1$, then $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$, so f is an increasing function.

→ if $0 < a < 1$, then $x_1 \cancel{>} x_2 \Rightarrow f(x_1) < f(x_2)$ So, f is a decreasing function.

4.> X-axis is a horizontal asymptote for the graph.

5.> The graph $y = a^x$ and $y = a^{-x}$ are reflections of each other about the y-axis.

6.> $a^{x+y} = a^x . a^y$ and $(a^x)^y = a^{xy}$

OR

Q.2 Explain the following term with suitable example.

(i) Singleton Set (ii) Disjoint Set

(iii) Partition of the set. (iv) Cardinality of A Finite set.

(v) Set of Sets.

Sol. (2)(i) Singleton Set -

→ Singleton Set or Singlet :- A set having only one element is known as a singleton set or simply singlet.

eg. $S := \{6\}$.

Sol. (2) (ii) Disjoint Set –

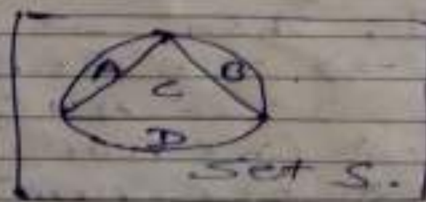
→ Disjoint Set :- Two set S and T are said to be disjoint when they have no element in common, i.e. when no element of S is in T and no element of T is in S.

eg. Let S be the set $\{a, b, c\}$ and T be the set $\{e, f, g\}$. Then S and T are disjoint.

Sol. (2) **(iii) Partition of the set –**

→ Partitions of set :– ~~A set~~ A set {A, B, C ...} of non-empty subsets of a set S is called the partition of S ~~if~~ if 1.) $A \cup B \cup C \cup \ldots\ldots = S$.

ii) ~~the $A \cap B \cap C \cap \ldots\ldots = \phi$~~ the intersection of every pair of distinct subsets is the empty, where the subset A, B, C . . . are called its member (element) or blocks.



eg. $A = \{3, 6, 9, 12, \ldots 24\}$

$B = \{1, 4, 7, 10, \ldots 25\}$

$C = \{2, 5, 8, 11, \ldots\ldots 23\}$.

$S = \{1, 2, 3, \ldots\ldots 25\}$

then $A \cup B \cup C = S$.

and $A \cap B = A \cap C = B \cap C = \phi$ so

{A, B, C} is a partition of set S.

Sol. (2) **(v) Set of Sets**.-

→ Set of Sets:-

A set itself ~~many~~ may sometimes be an element of another set. Then the latter set is called the set of sets.

Q.3 (a) Define the Comparable and Non-Comparable set with suitable example.

Sol. (3)(a)

→ Comparable Sets :- Two sets S and T are said to be comparable if $S \subset T$ or $T \subset S$. i.e. if one of the sets is a subset of the other.

eg. $S = \{1, 2, 3\}$ and $T = \{1, 2, 3, 4\}$. Then S is comparable to T, because $S \subset T$.

→ Non-Comparable Set :-

Two set S and T are said to be non-comparable if $S \not\subset T$ and $T \not\subset S$ i.e. none of the set is subset of each other.

eg. if $S = \{a, b\}$, $T = \{b, d, e\}$.

$S \not\subset T$ and $T \not\subset S$. the. S and T are not comparable.

Q.3 (b) What is the Function? Define the f-image, f-set and Representation of function by a diagram.

Sol. (3)(b)

Function –

→ Functions (or mapping) :-

Let A and B are two given sets. Let there exist a rule denoted by f, which associate to each element of A, a unique element of B. Then f is called a function or mapping from A to B. It is denoted by -

$$f : A \to B$$

which reads f is a function from A to B or f maps A to B.

f-image –

→ f-image :- Element b (∈B) corresponding to any element a (∈A) will be denoted by the symbol f(a) and is called f-image of a.

f-set –

→ f-set :- The set formed by all the f-image of the elements of A is called the image-set and is denoted by f(A).

Representation of function by a diagram-

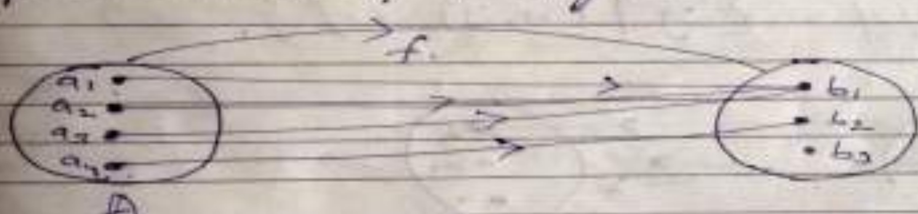
→ Representation by a diagram :-

The mapping or function $f : A \to B$ is said to be well defined if

1) every element $a \in A$ has an image $f(a)$ in B.
2) an element $a \in A$ has only one image $f(a)$ in B.

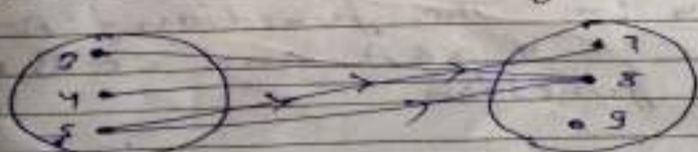
But, it is possible that two or more elements $a \in A$ (say $a_1, a_2, \ldots$) may have the same image in B. i.e. $f(a_1) = f(a_2) = f(a_3) \ldots$

The mapping or function $f : A \to B$ may be represented by a diagram.



eg. If $A = \{3, 4, 5\}$, $B = \{7, 8, 9\}$, $f(3) = 8$, $f(4) = 8$, $f(5) = 7$, $f(5) = 8$, find out whether it define a mapping.

Soln :- It does not define a mapping as the element 7, 8 of B are the images of the same element 5 of A.



OR

Q.3 (a) Explain the Matrix Representation of graph.

Find the Adjacency Matrix of the following relation.

X={ 1 , 2 , 3 } , Y={ 1 , 2,3 } and R={ (1,3) , (1,2) , (2,1) , (3,1) , (3,2) , (3,3) }.

Sol. (3)(a)

⇒ Adjacency Matrix of a Relation :-

Let elements of sets A and B be as

$A = \{a_1, a_2, \ldots a_m\}$

$B = \{b_1, b_2, \ldots b_n\}$

$R : A \to A$ then matrix of Relation is

$$M_R = [m_{ij}]_{m \times n} \qquad m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

M_R = matrix of Relation R or adjacency matrix of Relation

eg. Let $X = \{1,2,3\}$, $Y = \{1,2\}$ and $R = \{(2,1),(3,1),(3,2)\}$ then M_R is a 3×2 matrix, whose elements m_{21}, m_{31}, m_{32} are each equal to unity and other elements are all zero.

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

eg. Find the relation R, when

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$R : X \to Y$, where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3, y_4, y_5\}$

Sol^n then the relation is -

$R = \{(x_1, y_2), (x_1, y_3), (x_1, y_5), (x_2, y_1), (x_2, y_3)\}$.

Q.3 (b) Define the Join and Meet of Boolean Matrices. Also Perform the Join and Meet operation of following Two Boolean Matrices.

Sol. (3)(b)

→ Join and Meet of Boolean Matrices -

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two Boolean matrices of same size.

The join of A and B, denoted by $A \vee B$ is a matrix $C = [c_{ij}]_{m \times n}$

$$c_{ij} = \begin{cases} 1, & \text{if } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0, & \text{if } a_{ij} = 0 = b_{ij} \end{cases}$$

$[A \vee B = C]$

The meet of A and B, denoted By $A \wedge B$, is a matrix $D = [d_{ij}]_{m \times n}$ where

$$d_{ij} = \begin{cases} 1, & \text{if } a_{ij} = 1 = b_{ij} \\ 0, & \text{if } a_{ij} = 0 \text{ or } b_{ij} = 0 \end{cases}$$

$[A \wedge B = D]$

* The join and meet of two Boolean matrix are also Boolean matrices.

eg. compute the join and meet of the matrices.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$C = A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = A \wedge B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Q.4 (a) What is the function? Explain the Equal Function and Sum and Product of Function with suitable example.

Sol. (4)(a)

Function -

→ Functions (or mapping) :-

Let A and B are two given sets. Let there exist a rule denoted by f, which associate to each element of A, a unique element of B. Then f is called a function or mapping from A to B. It is denoted by -

$$f : A \to B$$

which reads f is a function from A to B or f maps A to B.

Equal Function –

⇒ Equal function :-

Two functions f and g are said to be equal iff -

1.) the domain of f = the domain of g.

2.) the co-domain of f = the co-domain of g.

3.) $f(x) = g(x)$ for every x belonging to their common domain.

eg. Let $A = \{1, 2\}$, $B = \{3, 6\}$

$f : A \to B$; $f(x) = x^2 + 2$

$g : A \to B$; $g(x) = 3x$

$$\left.\begin{array}{l} f(1) = 3 = g(1) \\ f(2) = 6 = g(2) \end{array}\right\} \Rightarrow f = g$$

Sum and Product of Function –

⇒ Sum and Product of Function

Let $f: A \to R$ and $g: B \to R$ be two function.

Then their sum $f+g$ and product fg can be defined as below.

$$(f+g)(x) = f(x) + g(x).$$

$$(fg)(x) = f(x) \cdot g(x).$$

$$\text{dom}(f+g) = \text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g).$$

eg. Let $f(x) = x^2$
$g(x) = \sqrt{x-1}$

where $\text{dom}(f) = (-\infty, \infty)$ and $\text{dom}(g) = [1, \infty)$
The obtain the sum and product of f and g and also find their domains.

Solⁿ :-

$$(f+g)(x) = f(x) + g(x)$$
$$= x^2 + \sqrt{x-1}$$

$$(fg)(x) = f(x) \cdot g(x).$$

$$= x^2 \cdot \sqrt{x-1}.$$

$$\text{dom}(f) \cap \text{dom}(g) = [1, \infty)$$

So, both $f(x)$ and $g(x)$ are defined only when $x \geq 1$.
Thus $\text{dom}(f+g) = \text{dom}(fg) = [1, \infty)$.

Q.4 (b) Explain the Logarithmic function and also define the Properties of Logarithmic function.

Sol. (4)(b)

→ 3.> Logarithmic Functions :-

Let $a(\neq 1) \in R^+$ and $x, y \in R$ such that $y = a^x$. Then x is called logarithm of y to the base a denoted by $\log_a y$. i.e

$$x = \log_a y \Leftrightarrow y = a^x$$

Thus, the function $f: R^+ \to R$ defined by

$$f(x) = \log_a x$$

is the logarithmic function with base a.

Since $a > 0$ so $y = a^x$ is always positive for every $x \in R$.

So we can say that $x = \log_a y$ is defined only when $y > 0$.

The domain of logarithmic function is $(0, \infty)$ and its range is $(-\infty, \infty)$.

Three bases of logarithms, namely e, 2 and 10 are often used.

$\log_e x = \ln x$ [natural logarithms]

$\log_2 x = \lg x$ [commonly used in Computer science]

$\log_{10} x = \log x$ [base 10 is usually omitted].

→ Properties of Logarithmic Functions.

Logarithmic function $f(x) = \log_a x$.

1.> $\mathrm{dom}(f) = (0, \infty)$, range $(f) = (-\infty, \infty)$.

2.> The x-intercept of the logarithmic graph 1.

3.> for $a > 1$, f is increasing.

for $0 < a < 1$, f is decreasing.

4.> Y-axis is a vertical asymptote of the graph.

5.> $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other and their graphs are mirror image of each-other about the line $y = x$.

6.> The graphs of $y = \log_a x$ and $y = \log_{1/a} x$ are reflection of each-other about the X-axis.

Theorem - Let $a(\neq 1)$, $b(\neq 1)$, x and y are any positive real number and $m, n \in R$. Then.

OR

Q.4 (a) Explain the Absolute Value Function and Characteristic function with suitable example.

Sol. (4)(a)

Absolute Value Function -

→ 4) Absolute Value Function :-

The function $f : R \to R$ defined by

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is called absolute value function.

The graph of the function $y = |x|$ is as shown.

dist b/w −2 to −4 is $|-2-(-4)| = |-2+4| = 2$

dist b/w −2 to 5 is $|-2-5| = |-7| = 7$.

eg. $|3| = 3$; $|-8| = 8$; eg $|-2| = -(-2) = 2$

the absolute value fn is used to measure the dist. b/w two no. dist b/w x and 0 is $|x-0|$ i.e. $|x-0|$

Characteristic function -

→ 9) Characteristic function :-

Let A be a subset of the Universal set $U = \{u_1, u_2, u_3, \ldots, u_n\}$. The characteristic function of A is defined as a function from U to {0, 1}. i.e.

$f_A : U \to \{0, 1\}$.

by the rule $f_A(u_i) = \begin{cases} 0, & \text{if } u_i \notin A \\ 1, & \text{if } u_i \in A \end{cases}$

This function is used in representing sets in a computer.

Hence f_A can be represented by a sequence of 0's and 1's of length equal to the cardinality of U because $0 \notin A$ and $1 \in A$.

eg. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{3, 4\}$, $B = \{4, 6, 8\}$, $C = \{6, 8, 10\}$

then $f_A(x) = \begin{cases} 1, & \text{when } x = 3, \text{ or } 4 \\ 0, & \text{if } x \neq 3, \text{ or } 4 \end{cases}$

Thus.

f_A corresponds to sequence = 0, 0, 1, 1, 0, 0, 0, 0, 0, 0.

f_B corresponds to sequence = 0, 0, 0, 1, 0, 1, 0, 1, 0, 0.

f_C corresponds to sequence = 0, 0, 0, 0, 0, 1, 0, 1, 0, 1.

Q.4 (b) Define the following term with suitable example.

Sol. (4)(b) (i) Bijection Mapping

→ 5.) Bijection Mapping :- A mapping $f : A \to B$ is said to be a Bijection if it is one-one as well as onto mapping.

$f : A \to B$ is Bijection if $f(a) = f(b) \Rightarrow a = b$ and $f(A) = B$.

eg. The mapping $f : Z \to Z$, $f(n) = n+1$ is a bijection because for any $x_1, x_2 \in Z$

$x_1 \neq x_2 \Rightarrow x_1 + 1 \neq x_2 + 1 \Rightarrow f(x_1) \neq f(x_2)$

$\Rightarrow f$ is one-one and $f(Z) = Z \Rightarrow f$ is onto.

Sol. (4)(b) (ii) One-One Mapping.

→ 3.) One-One Mapping (or Injective Mapping) :- A mapping $f : A \to B$ is said to be a one-one mapping if different element of A have different f-image in B. Thus,

$f : A \to B$ is one-one $\Leftrightarrow f(a) = f(b) \Rightarrow a = b, \; \forall a, b \in A$

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$

eg. let I be the set of integers and B the set of all even integers then the mapping $f : I \to B$, defined by $f(x) = 2x$, $x \in I$ is an into mapping which is also one-one.

eg. let $f : Z \to Z$, $f(x) = -x$. Then f is an injection because for any $a, b \in Z$, $a \neq b \Rightarrow -a \neq -b \Rightarrow f(a) \neq f(b)$.

Sol. (4)(b) (iii) Onto Mapping.

→ 2.) Onto Mapping (or Surjective Mapping):- If the mapping $f: A \to B$ is such that each element of B is the f-image of at least one element of A, then f is called onto mapping.

$$f(A) = B$$

eg. let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16\}$ and $f(x) = x^2$, $\forall x \in A$, then $f: A \to B$ is an onto function as every element of B has a pre-image in A under f.